Deferential equations modelling for car motion and dynamics

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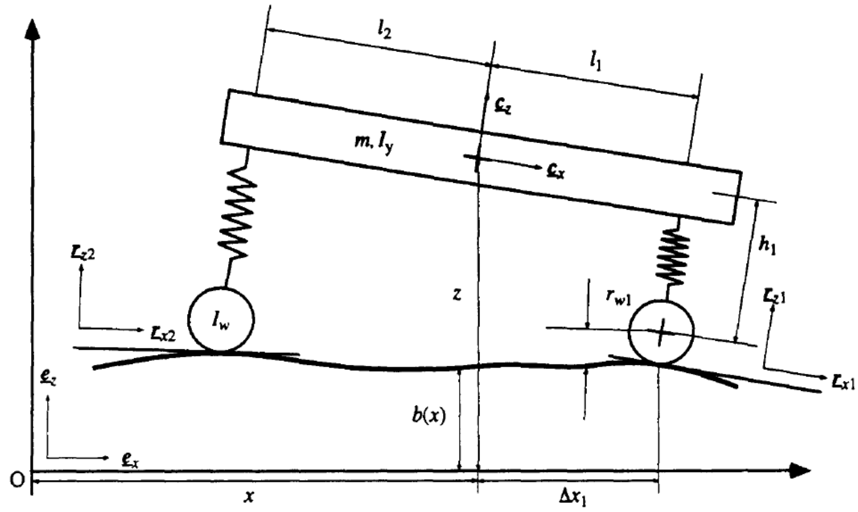
N# BD7DQV

1-nonlinear longitudinal model

1-1 reference frames



The figure above for vehicle longitudinal model showing the coordinates with tow reference frames first one fived with the earth (e) the second one fixed with car (c) where we can see the axes (x & z) and the axes y points to the paper.

The vehicle frame is created by rotation of earth frame around y axis with pitch angle (θ) then the coordinate transformation matrices are

$$\begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} = \begin{pmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \qquad (1)$$

1-2 rotational equation of motion:

earth relative car angular velocity is given by the equation:

$$w = \dot{\theta} e_y = w_y e_y \qquad (2)$$

And the angular acceleration is given by:

$$\dot{w} = \dot{w}_y e_y \qquad (3)$$

1-3 translational motion equation

the position vector of car centre of mass from the origin of earth frame is given by

$$\underline{P}_{\mathrm{CM}} = x\underline{e}_x + z\underline{e}_z \tag{4}$$

In this case we can calculate the velocity of mass centre by the equations:

$$\underline{v}_{\mathrm{CM}} = \dot{x}\underline{e}_x + \dot{z}\underline{e}_z \tag{5}$$

$$= v_x\underline{c}_x + v_z\underline{c}_z \tag{6}$$

Appling the equation (1) to the equation (6) we can find:

$$\underline{v}_{\mathrm{CM}} = (v_x \cos\theta + v_z \sin\theta)\,\underline{e}_x + (-v_x \sin\theta + v_z \cos\theta)\,\underline{e}_z \tag{7}$$

And from both (5) & (7) we get :

$$\dot{x} = v_x \cos\theta + v_z \sin\theta \tag{8}$$

$$\dot{z} = -v_x \sin\theta + v_z \cos\theta \tag{9}$$

By differentiating of equation (6) we get the acceleration of mass centre by

$$\begin{aligned} \underline{a} &= \dot{v}_x\underline{c}_x + \dot{v}_z\underline{c}_z + \omega_y\underline{c}_y \times (\dot{v}_x\underline{c}_x + \dot{v}_z\underline{c}_z) \\ &= (\dot{v}_x + \omega_y v_z)\,\underline{c}_x + (\dot{v}_z - \omega_y v_x)\,\underline{c}_z \end{aligned} \tag{10}$$

By knowing the external forces affecting on the vehicle according to the general equation:

$$\underline{F} = F_x\underline{c}_x + F_z\underline{c}_z$$

And depending on Newton's second law we can define the final motion equations as following:

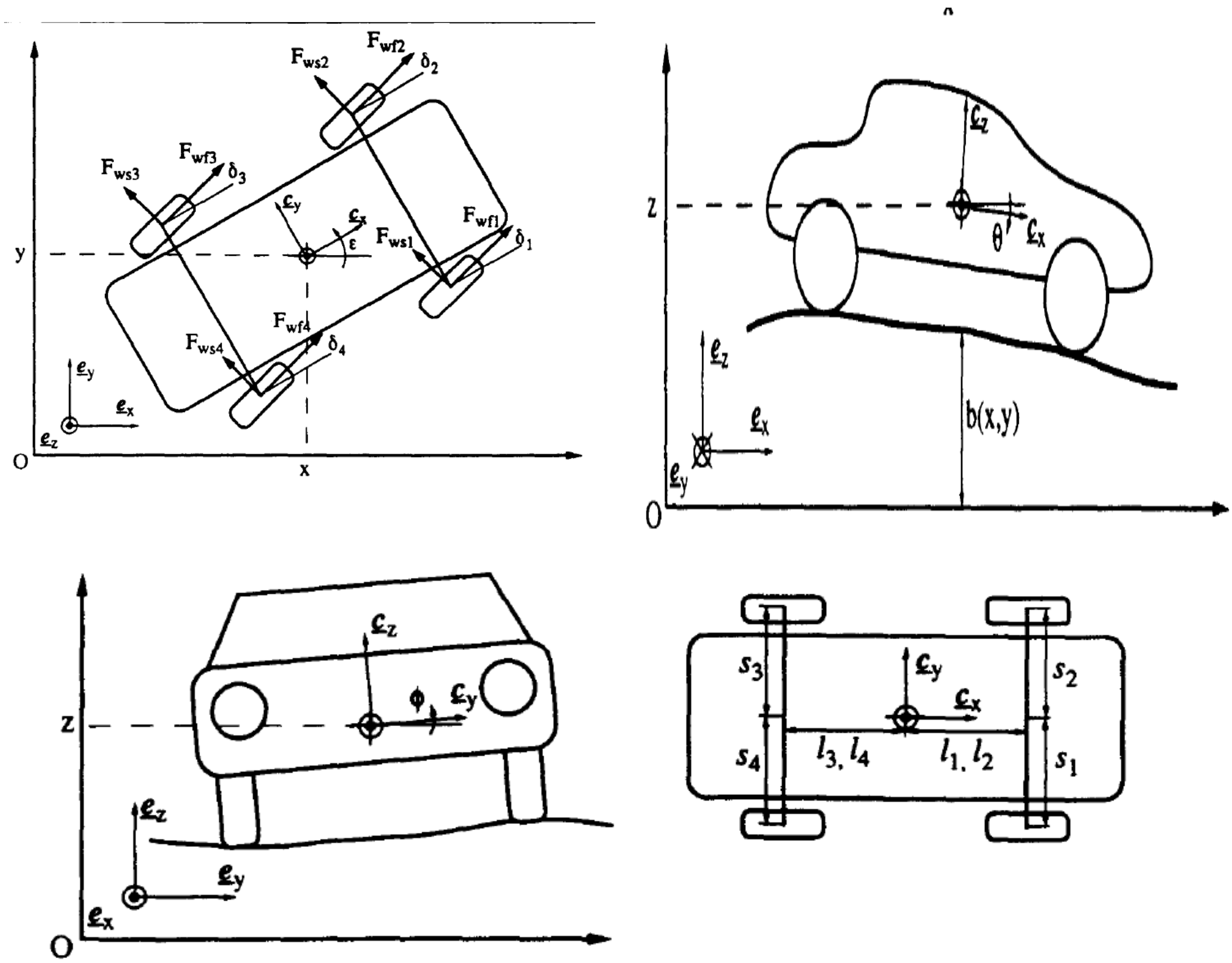
$$\dot{v}_x = -\omega_y v_z + \frac{F_x}{m} \tag{11}$$

$$\dot{v}_z = \omega_y v_x + \frac{F_z}{m} \tag{12}$$

Where

M	The mass of car
Fx	External forces according to axis x
Fy	External forces according to axis z

2- nonlinear lateral & longitudinal model:



Same like previous we have frame fixed to the earth(e) and another to the car (c) according to the figures we can define the following angles like:

θ	Pitch angle
ξ	Yaw angle
ϕ	Roll angle

In this case we have transformation matrices as following:

$$\begin{bmatrix} \underline{c}_x \\ \underline{c}_y \\ \underline{c}_z \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\varepsilon & \sin\varepsilon & 0 \\ -\sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{e}_x \\ \underline{e}_y \\ \underline{e}_z \end{bmatrix} \tag{13}$$

2-2 rotational equation of motion:

With the angular rotations defined above, the angular velocity of the vehicle is given by:

$$\underline{\omega} = \dot{\varepsilon}\underline{e}_z + \dot{\phi}\underline{a}_x + \dot{\theta}\underline{b}_y \tag{14}$$

Use the coordinate transformation matrices (13) to obtain the vehicle

angular velocity in the vehicle-fixed coordinate frame as

$$\begin{aligned} \underline{\omega} &= (\dot{\phi}\cos\theta - \dot{\varepsilon}\cos\phi\sin\theta)\underline{c}_x + (\dot{\theta} + \dot{\varepsilon}\sin\phi)\underline{c}_y + (\dot{\phi}\sin\theta + \dot{\varepsilon}\cos\phi\cos\theta)\underline{c}_z \\ &= \omega_x\underline{c}_x + \omega_y\underline{c}_y + \omega_z\underline{c}_z \end{aligned} \tag{15}$$

then rotational kinematic equations of motion are:

$$\dot{\varepsilon} = \frac{1}{\cos\phi}(-\sin\theta\omega_x + \cos\theta\omega_z) \tag{16a}$$

$$\dot{\phi} = \cos\theta\omega_x + \sin\theta\omega_z \tag{16b}$$

$$\dot{\theta} = \tan\phi(\sin\theta\omega_x - \cos\theta\omega_z) + \omega_y \tag{16c}$$

2-3 Translational Equations of Motion

the position vector from the Earth-fixed origin 0 to the vehicle centre of mass:

$$\underline{P}_{\mathrm{CM}} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z$$

Then the velocity of the mass center can be expressed either in Earth-fixed or vehicle-fixed coordinates as:

$$\underline{v}_{\mathrm{CM}} = \dot{x}\underline{e}_x + \dot{y}\underline{e}_y + \dot{z}\underline{e}_z \tag{17}$$

$$= \quad v_x \underline{c}_x + v_y \underline{c}_y + v_z \underline{c}_z \tag{18}$$

Applying Equation (13) to transform Equation (18) into an Earth-fixed frame leads to:

$$\begin{aligned}
\underline{v}_{\mathrm{CM}} = \; & [v_x(\cos\varepsilon\cos\theta - \sin\varepsilon\sin\phi\sin\theta) - v_y\sin\varepsilon\cos\phi + v_z(\cos\varepsilon\sin\theta + \sin\varepsilon\sin\phi\cos\theta)]\,\underline{e}_x + \\
& [v_x(\sin\varepsilon\cos\theta + \cos\varepsilon\sin\phi\sin\theta) + v_y\cos\varepsilon\cos\phi + v_z(\sin\varepsilon\sin\theta - \cos\varepsilon\sin\phi\cos\theta)]\,\underline{e}_y + \\
& [-v_x\cos\phi\sin\theta + v_y\sin\phi + v_z\cos\phi\cos\theta]\,\underline{e}_z
\end{aligned} \tag{8.71}$$

Hence the translational kinematic equations follow immediately from (18) and (19).

$$\begin{aligned}
\dot{x} \;=\; & v_x(\cos\varepsilon\cos\theta - \sin\varepsilon\sin\phi\sin\theta) - v_y\sin\varepsilon\cos\phi + \\
& v_z(\cos\varepsilon\sin\theta + \sin\varepsilon\sin\phi\cos\theta)
\end{aligned} \tag{20a}$$

$$\begin{aligned}
\dot{y} \;=\; & v_x(\sin\varepsilon\cos\theta + \cos\varepsilon\sin\phi\sin\theta) + v_y\cos\varepsilon\cos\phi + \\
& v_z(\sin\varepsilon\sin\theta - \cos\varepsilon\sin\phi\cos\theta)
\end{aligned} \tag{20b}$$

$$\dot{z} \;=\; -v_x\cos\phi\sin\theta + v_y\sin\phi + v_z\cos\phi\cos\theta \tag{20c}$$

The acceleration of the vehicle centre of mass can be found by differentiating (18).

$$\begin{aligned}
\underline{a} \;&=\; \dot{v}_x\underline{c}_x + \dot{v}_y\underline{c}_y + \dot{v}_z\underline{c}_z + (\omega_x\underline{c}_x + \omega_x\underline{c}_x + \omega_x\underline{c}_x) \times (\omega_x\underline{c}_x + \omega_x\underline{c}_x + \omega_x\underline{c}_x) \\
&=\; (\dot{v}_x + \omega_y v_z - \omega_z v_y)\underline{c}_x + (\dot{v}_y + \omega_z v_x - \omega_x v_z)\underline{c}_y + (\dot{v}_z + \omega_x v_y - \omega_y v_x)\underline{c}_z
\end{aligned} \tag{21}$$

If the total external force applied to the vehicle is known,

$$\underline{F} = F_x \underline{c}_x + F_y \underline{c}_y + F_z \underline{c}_z$$

the translational dynamic equations are obtained from Newton's second law,

$$\dot{v}_x = \omega_z v_y - \omega_y v_z + \frac{F_x}{m} \tag{22a}$$

$$\dot{v}_y = \omega_x v_z - \omega_z v_x + \frac{F_y}{m} \tag{22b}$$

$$\dot{v}_z = \omega_y v_x - \omega_x v_y + \frac{F_z}{m} \tag{22c}$$